On the hydrodynamic stability of two viscous incompressible fluids in parallel uniform shearing motion

By SAUL FELDMAN
AVCO Research Laboratory, Everett, Massachusetts

(*Received* 15 *November* 1956)

SUMMARY

A new problem in hydrodynamic stability is investigated. Given two contiguous viscous incompressible fluids, the fluid on one side of the plane interface being bounded by a solid wall and that on the other side being unbounded, the problem is to determine the hydrodynamic stability when the fluids are in steady unidirectional motion, parallel to the interface, with uniform rate of shear in each fluid. The mathematical analysis, based on small disturbance theory, leads to a characteristic value problem in a system of two linear ordinary differential equations. The essential dimensionless parameters that appear in the present problem are the viscosity ratio m, the density ratio r, the Froude number F, and the Weber number W, as well as the parameters α, R (which is proportional here to the flow rate of the inner fluid) and c, that occur in the study of hydrodynamic stability of a single fluid. The results obtained are presented graphically for most fluid combinations of possible interest. The neutral stability curve in the (α, R)-plane is single-looped, as in the boundary layer case. The calculated critical Reynolds numbers are higher than the values observed in liquid film cooling experiments. (In these experiments, the outer fluid is usually a turbulent gas, in which the thickness of the laminar sublayer is of the same order of magnitude as the liquid film thickness.) General agreement between the theoretical and experimental values exists for all critical quantities except the Reynolds number. Gravity and surface tension are found here to have a destabilizing effect on the flow, in agreement with experimental evidence. Semi-infinite plane Couette flow is a special case of the present problem and the known stability of this flow is recovered. The linear velocity profile of two adjacent fluids with the same viscosity, but different densities, is shown to be unstable for high enough Reynolds numbers. The Reynolds stress distribution for a neutral oscillation in the general case is discussed qualitatively.

I. INTRODUCTION

The interest in the present problem stems from the desire to understand what happens when a liquid film flows over a flat surface, dragged along by a high speed gas. For certain liquid flow rates, the film surface becomes wavy, and detached parcels of liquid from the main body are entrained by the gas and carried downstream. This situation arises, among numerous engineering applications, in connection with film cooling of a solid boundary, as reported by Knuth (1954).

A classical problem related to the one mentioned above is the generation of ocean waves by wind. This problem has recently been investigated theoretically by Lock (1954), who was the first to have included in the analysis all the physical properties of the air and water. His calculations are incomplete, and the results obtained are quite different from the usual single-fluid results.

The problem of the stability of stratified motion of different fluids has been studied by Taylor (1931) and Goldstein (1931), who did not include viscosity in their analyses. Taylor investigated continuous and discontinuous density and velocity distributions. Goldstein treated similar problems, his investigations being a generalization to heterogeneous stratified fluids of Rayleigh's (1887) work on the homogeneous case.

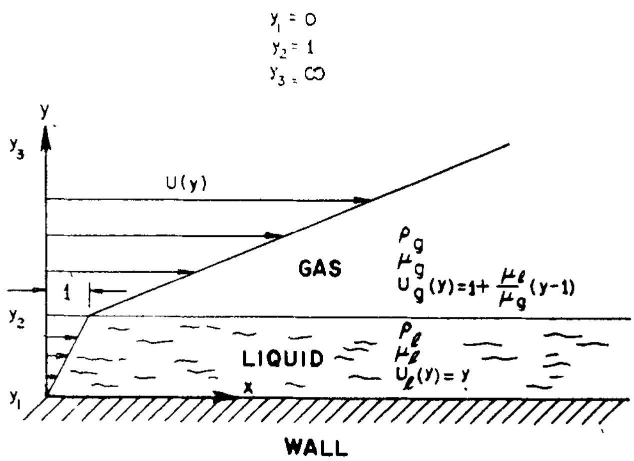

Figure 1. Undisturbed velocity profile to be investigated.

The problem of liquid film stability has been investigated experimentally by Kinney, Abramson & Sloop (1952) and by Knuth (1954), who were concerned with liquid film cooling applications where the gas stream was always turbulent. York, Stubbs & Teck (1953) studied the mechanism of disintegration of liquid sheets experimentally, and they proposed an inviscid model based on an extension of Lamb's work (1932).

The aim of this investigation is to solve the hydrodynamic stability problem when two viscous incompressible fluids, in two-dimensional, laminar, uniform shearing motion (figure 1) are perturbed by a small arbitrary disturbance. One of the fluids, from now on called the liquid, is bounded in the direction normal to the flow by a solid wall and by the second fluid, called the gas, of semi-infinite extent in the direction normal to the flow.

By 'solving the problem' is meant finding the relationships satisfied by certain parameters when neutral stability exists. This implies that a neutral-stability hypersurface, whose coordinates are the physical variables involved, could be constructed that would separate regions of stability and instability. In other words, if the physical properties of the fluids are given, we wish to determine the minimum critical Reynolds number* at which instability begins.

It will be realized from the above remarks that the chosen model is an idealization of the physical situation. The most serious criticism is that the gas motion, in the actual case, is turbulent in most cases of interest, and the velocity profile is not a linear function of the distance away from the wall. Some justification for the present approach lies in the fact that, at least in the laminar sublayer of the turbulent flow, the gas is laminar with an almost linear velocity profile. (The ratio of gas laminar sublayer thickness to liquid film thickness in existing experiments is of the order of unity.) There is still one other reason for the approach used (besides the obvious one of greater tractability when the turbulence is ignored), for it has been definitely shown by Zondek & Thomas (1953) that semi-infinite plane Couette flow is always stable. Now, this situation is a special case of the present model, and it is interesting in this connection to know how a discontinuity in density or viscosity affects the stability of uniform shearing motion.

II. The boundary value problem and its solution

1. *The Orr–Sommerfeld differential equation and its general solution for plane Couette flow*

The task now is to formulate mathematically the problem of stability of two-dimensional laminar motion. This has been done in the past by numerous authors (e.g. Lin 1945). For the sake of completeness, however, a brief description of the derivation of the disturbance equation will be given here.

Let all coordinates and velocities be made dimensionless by the use of a reference length δ, and a reference velocity $\overline{U}_2$. Consider a basic flow in the x-direction with a velocity profile $U(y)$. The Navier–Stokes and continuity equations in the xy-plane can be perturbed by assuming the velocities in the x- and y-directions and the pressure to have the form

$$U(y)+u(x,y,t), \qquad v(x,y,t), \qquad P+p(x,y,t), \tag{1}$$

where the lower-case symbols indicate small quantities, and t is the dimensionless time (i.e. time $\times\, \overline{U}_2/\delta$). The introduction of (1) into the Navier–Stokes equations leads to two linear partial differential equations from which the pressure can be eliminated by cross differentiation and subtraction. The result is a linear partial differential equation containing

* Defined, for given flow physical properties, as the smallest value of R for which incipient disturbances will become amplified, i.e. the smallest value of R of the neutral stability curve in the (α, R)-plane.

u and v as the dependent variables, with x, y and t as the independent variables. The equation of continuity guarantees the existence of a stream function $\psi(x, y, t)$, such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x},$$

and which, when used in the differential equation last mentioned, reduces it to a linear partial differential equation in terms of only one dependent variable $\psi(x, y, t)$. The fact that this equation is linear is very important, since this means that superposition of solutions is allowable. This implies that if any arbitrary disturbance is decomposed into its Fourier components, it is then sufficient to solve the problem for one general sinusoidal oscillation. After the solution is obtained, it will be necessary to consider all possible frequencies and see how they affect the behaviour of the solution.

In order to separate the variables in the partial differential equation for ψ, let

$$\psi(x, y, t) = \phi(y) e^{i\alpha(x-ct)}, \tag{2}$$

where α is the wave number, assumed positive without any loss of generality, and c is the complex wave velocity which may be expressed as

$$c = c_r + ic_i, \tag{3}$$

where c_r is the wave velocity and c_i allows for amplification of disturbances if $c_i > 0$, damping of disturbances if $c_i < 0$, and neutral disturbances if $c_i = 0$. The partial differential equation for ψ then reduces to the Orr–Sommerfeld equation

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = -\frac{i}{\alpha R}(\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi), \tag{4}$$

where the primes indicate derivatives with respect to y, and R is the Reynolds number $\rho\delta \bar{U}_2/\mu$ (ρ is the density of the fluid and μ the viscosity).

If the velocity profile of the undisturbed flow is a linear function of y, as in the case of interest, $U'' = 0$, and (4) becomes

$$(U-c)(\phi'' - \alpha^2\phi) = -\frac{i}{\alpha R}(\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi), \tag{5}$$

which is a linear fourth-order total differential equation in the complex y-plane. As first pointed out by Lin (1945), equation (5) has four linearly independent solutions, which are analytic functions of y and entire functions of the parameters c, α and αR. We will now solve (5). A transformation which was first pointed out by Orr (1906) and later independently by Sommerfeld (1909) is

$$\phi''(y) - \alpha^2\phi(y) = \zeta(y), \tag{6}$$

which when used in (5) yields

$$\zeta'' - [i\alpha R(U-c) + \alpha^2]\zeta = 0. \tag{7}$$

Now, let

$$\zeta(y) = h(z), \qquad z = \left\{\frac{\alpha^2}{(\alpha R)^{2/3}} + i(\alpha R)^{1/3}[U(y) - c]\right\}[U'(y)]^{-2/3}, \tag{8}$$

so that (7) can be written as

$$h''(z) + zh(z) = 0, \tag{9}$$

which is the so-called Stokes equation. The solution can be obtained in terms of contour integrals by using Laplace's method (Morse & Feshbach 1953, pp. 582–585). The result is

$$h(z) = 2k_3 h_1(z) + 2k_4 h_2(z), \tag{10}$$

where the factor 2 is introduced for convenience in later calculations, k_3 and k_4 are arbitrary constants, and

$$h_1(z) = \frac{k}{i\pi}\int_{L_1} \exp(zt + \tfrac{1}{3}t^3)\,dt, \qquad k = (12)^{1/6}e^{-i\pi/6}, \tag{11}$$

$$h_2(z) = -\frac{k^*}{i\pi}\int_{L_2} \exp(zt + \tfrac{1}{3}t^3)\,dt, \quad k^* = (12)^{1/6}e^{-i\pi/6}, \tag{12}$$

in which L_1 and L_2 are the contours of integration shown in figure 2, h_1 and h_2 are entire functions of z, and t is a complex variable of integration.

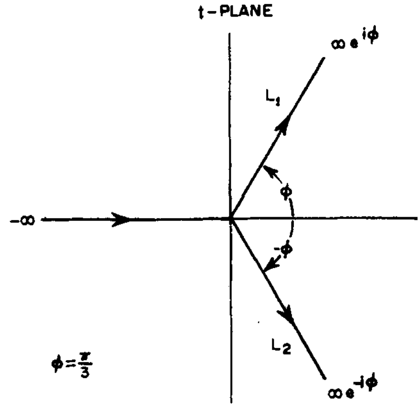

Figure 2. Paths for contour integrals in the solution of Stokes' equation.

The functions $h_1(z)$ and $h_2(z)$ may also be written in terms of Hankel functions of order one-third as

$$h_1(z) = (\tfrac{2}{3}z^{3/2})^{1/3}H^{(1)}_{1/3}(\tfrac{2}{3}z^{3/2}), \qquad h_2(z) = (\tfrac{2}{3}z^{3/2})^{1/3}H^{(2)}_{1/3}(\tfrac{2}{3}z^{3/2}). \tag{13}$$

It will be recalled (see, for instance, Copson (1935)) that the Hankel functions are of oscillatory nature, $H^{(1)}_{1/3}$, the function of the first kind, being damped exponentially as $|\tfrac{2}{3}z^{3/2}|$ becomes large, while $H^{(2)}_{1/3}$ increases exponentially under the same conditions.

From (8) and (10) we have

$$\zeta(y) = 2k_3\zeta_1(y) + 2k_4\zeta_2(y),$$

where

$$\zeta_1(y) = h_1(z), \qquad \zeta_2(y) = h_2(z), \tag{14}$$

which when inserted in (6) gives

$$\phi''(y) - \alpha^2\phi(y) = 2k_3\zeta_1(y) + 2k_4\zeta_2(y). \tag{15}$$

The complete solution of (5) is then

$$\phi(y) = k_1\phi_1(y) + k_2\phi_2(y) + k_3\phi_3(y) + k_4\phi_4(y), \tag{16}$$

where
$$\left.\begin{aligned}
\phi_1(y) &= e^{-\alpha y}, \qquad \phi_2(y) = e^{\alpha y},\\
\phi_3(y) &= \frac{1}{\alpha}\left\{\int^{y} e^{\alpha(y-t)}\zeta_1(t)\,dt - \int^{y} e^{-\alpha(y-t)}\zeta_1(t)\,dt\right\},\\
\phi_4(y) &= \frac{1}{\alpha}\left\{\int^{y} e^{\alpha(y-t)}\zeta_2(t)\,dt - \int^{y} e^{-\alpha(y-t)}\zeta_2(t)\,dt\right\}.
\end{aligned}\right\} \tag{17}$$

t is a variable of integration and should not be confused with the notation used to denote time. ζ_1 and ζ_2 are given by (14).

2. *Differential equations for the present problem*

When the phenomenon under investigation involves the stratified motion of two fluids, an equation like (5) must be used for each fluid, with the proper boundary conditions at the interface. Let δ and $\overline{U}_2$ as used to render (5) dimensionless be, respectively, the height of the liquid sheet and its surface velocity. Lower-case and capital Greek letters will represent conditions in the liquid and gas respectively, and the subscripts l and g will denote quantities evaluated in the liquid and in the gas. We then choose

$$\psi = \phi(y)e^{i\alpha(x-ct)}, \qquad \Psi = \Phi(y)e^{i\alpha(x-ct)} \tag{18}$$

as our representation of the disturbance stream function in the liquid and gas respectively. The disturbance velocities then become

$$\left.\begin{aligned}
u_l &= \frac{\partial\psi}{\partial y} = \phi'(y)e^{i\alpha(x-ct)}, & u_g &= \frac{\partial\Psi}{\partial y} = \Phi'(y)e^{i\alpha(x-ct)},\\
v_l &= -\frac{\partial\psi}{\partial x} = -i\alpha\phi(y)e^{i\alpha(x-ct)}, & v_g &= -\frac{\partial\Psi}{\partial x} = -i\alpha\Phi(y)e^{i\alpha(x-ct)},
\end{aligned}\right\} \tag{19}$$

while the corresponding Orr–Sommerfeld equations are

$$(U_l-c)(\phi''-\alpha^2\phi) = -\frac{i}{\alpha R_l}(\phi^{\text{iv}}-2\alpha^2\phi''+\alpha^4\phi), \qquad (0 \leqslant y \leqslant 1), \tag{20}$$

$$(U_g-c)(\Phi''-\alpha^2\Phi) = -\frac{i}{\alpha R_g}(\Phi^{\text{iv}}-2\alpha^2\Phi''+\alpha^4\Phi), \quad (1 \leqslant y \leqslant \infty), \tag{21}$$

where
$$R_l = \frac{\rho_l\,\delta\overline{U}_2}{\mu_l}, \qquad R_g = \frac{\rho_g\,\delta\overline{U}_2}{\mu_g}, \qquad U_l = y. \tag{22}$$

Equating the shear stresses at the interface of the basic flow gives

$$U_g = 1+\frac{\mu_l}{\mu_g}(y-1). \tag{23}$$

The solutions of (20) and (21) can now be written, from (16), as

$$\phi(y) = k_1\phi_1(y)+k_2\phi_2(y)+k_3\phi_3(y)+k_4\phi_4(y), \qquad (0 \leqslant y \leqslant 1), \tag{24}$$

$$\Phi(y) = K_1\Phi_1(y)+K_2\Phi_2(y)+K_3\Phi_3(y)+K_4\Phi_4(y), \quad (1 \leqslant y \leqslant \infty), \tag{25}$$

where the ϕ's are given by (17), and the Φ's can also be obtained from the same equation by using the appropriate value of z in the gas, given by (8).

3. *Boundary conditions*

The boundary conditions will be written in terms of dimensionless variables.

Let subscripts 1, 2 and 3 used with the coordinate y denote, respectively, the wall, interface and infinity, i.e. $y_1 = 0$, $y_2 = 1$ and $y_3 = \infty$.

At the wall ($y = y_1 = 0$), both components of the disturbance velocities must vanish; i.e.

$$v_l(y_1) = 0, \qquad u_l(y_1) = 0. \tag{26}$$

At the interface ($y = y_2 = 1$), the following conditions must hold:

(a) Both fluids move together with no vacuum layer between them; thus

$$v_l(y_2) - v_g(y_2) = 0. \tag{27}$$

(b) There is no slip between the fluids in the direction of flow; thus

$$u_l(y_2) - u_g(y_2) = 0. \tag{28}$$

(c) The shear stress must be continuous; thus

$$u_l\left(\frac{\partial v_l}{\partial x} + \frac{\partial u_l}{\partial y}\right) - \mu_g\left(\frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y}\right) = 0. \tag{29}$$

(d) Because of surface tension, the normal stress is discontinuous according to the relation

$$p_T - \sigma_{yyg} = -\sigma_{yyl},$$

where p_T is the effective pressure caused by surface tension. If σ is the surface tension coefficient and l_r the radius of curvature of the interface, p_T can be written as

$$p_T = \sigma/l_r,$$

where
$$\frac{1}{l_r} = -\frac{d^2\bar{y}}{d\bar{x}^2}\bigg/\left[1+\left(\frac{d\bar{y}}{d\bar{x}}\right)^2\right]^{3/2},$$

the negative sign having been chosen because p_T is to be positive when $d^2\bar{y}/d\bar{x}^2$ is negative. The normal force equation can be rewritten as

$$p_g - 2\mu_g\frac{\partial v_g}{\partial y} - p_l + 2\mu_l\frac{\partial v_l}{\partial y} + p_T = 0. \tag{30}$$

At infinity ($y = y_3 = \infty$), the disturbance must vanish; i.e.

$$v_g(y_3) = 0, \qquad u_g(y_3) = 0. \tag{31}$$

Equations (26) to (31) are the eight boundary conditions necessary for solving the system of two-fourth order differential equations given by (24) and (25).

4. *The secular equation*

Equations (13), (14) and (17) show that when $y \to \infty$, $\Phi_2(y)$ and $\Phi_4(y) \to \infty$. In order to satisfy the boundary conditions given in (31), it is necessary that in (25)

$$K_2 = K_4 = 0, \tag{32}$$

which when used with (19) allows the boundary conditions (26) to (31) to be rewritten as

$$k_1\phi_{11}+k_2\phi_{21}+k_3\phi_{31}+k_4\phi_{41} = 0, \tag{33}$$

$$k_1\phi'_{11}+k_2\phi'_{21}+k_3\phi'_{13}+k_4\phi'_{41} = 0, \tag{34}$$

$$k_1\phi_{12}+k_2\phi_{22}+k_3\phi_{32}+k_4\phi_{42}-K_1\Phi_{12}-K_3\Phi_{32} = 0, \tag{35}$$

$$k_1\phi'_{12}+k_2\phi'_{22}+k_3\phi'_{32}+k_4\phi'_{42}-K_1\Phi'_{12}-K_3\Phi'_{32} = 0, \tag{36}$$

$$\begin{aligned}k_1(\alpha^2\phi_{12}+\phi''_{12})+k_2(\alpha^2\phi_{22}+\phi''_{22})&+\\ +k_3(\alpha^2\phi_{32}+\phi''_{32})+k_4(\alpha^2\phi_{42}+\phi''_{42})&+\\ -\frac{K_1}{m}(\alpha^2\Phi_{12}+\Phi''_{12})-\frac{K_3}{m}(\alpha^2\Phi_{32}+\Phi''_{32})&=0,\end{aligned} \tag{37}$$

$$\begin{aligned}&k_1\left\{\left[\frac{\alpha^2}{W}+\frac{1}{F(1-c)}+1\right]R\phi_{12}-[(1-c)R-i3\alpha]\phi'_{12}-\frac{i}{\alpha}\phi'''_{12}\right\}+\\ &+k_2\left\{\left[\frac{\alpha^2}{W}+\frac{1}{F(1-c)}+1\right]R\phi_{22}-[(1-c)R-i3\alpha]\phi'_{22}-\frac{i}{\alpha}\phi'''_{22}\right\}+\\ &+k_3\left\{\left[\frac{\alpha^2}{W}+\frac{1}{F(1-c)}+1\right]R\phi_{32}-[(1-c)R-i3\alpha]\phi'_{32}-\frac{i}{\alpha}\phi'''_{32}\right\}+\\ &+k_4\left\{\left[\frac{\alpha^2}{W}+\frac{1}{F(1-c)}+1\right]R\phi_{42}-[(1-c)R-i3\alpha]\phi'_{42}-\frac{i}{\alpha}\phi'''_{42}\right\}-\\ &-\frac{K_1}{m}\left\{\left[m+\frac{1}{F(1-c)}\right]Rrm\Phi_{12}-[(1-c)Rrm-i3\alpha]\Phi'_{12}-\frac{i}{\alpha}\Phi''_{12}\right\}-\\ &-\frac{K_3}{m}\left\{\left[m+\frac{1}{F(1-c)}\right]Rrm\Phi_{32}-[(1-c)Rrm-i3\alpha]\Phi'_{32}-\frac{i}{\alpha}\Phi'''_{32}\right\}=0,\end{aligned} \tag{38}$$

where $\phi_{11}\equiv\phi_1(y_1)$, $\phi_{23}\equiv\phi_2(y_3)$, etc., and

$$m=\mu_l/\mu_g;\qquad r=\rho_g/\rho_l;\qquad R\equiv R_l;\qquad F=\overline{U}_2^2/g\delta;\qquad W=\rho_l\delta\overline{U}_2^2/\sigma. \tag{39}$$

R_g does not enter in the relationships of this section because it is clear, from (25) and (26), that

$$R_g = R_l rm.$$

For the details of the derivation of (38), reference may be made to the author's thesis (Feldman 1955, Appendix A).

Thus, the essential dimensionless parameters of the problem have been defined as α, R, c, m, r, F and W, where F and W denote, respectively, the Froude and Weber numbers. It is seen that R_g has disappeared from the problem as an explicit parameter. This seems to be in line with experimental results (Knuth 1954, pp. 362–363), which show that the inception point of unstable disturbances is independent of the gas-stream Reynolds number.

If the preceding set of equations, (33) to (38), is to have a non-trivial solution for the k's and K's, the following relation, the so-called secular equation, must hold:

$$\begin{vmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} & 0 & 0 \\ \phi'_{11} & \phi'_{21} & \phi'_{31} & \phi'_{41} & 0 & 0 \\ \phi_{12} & \phi_{22} & \phi_{32} & \phi_{42} & -\Phi_{12} & -\Phi_{32} \\ \phi'_{12} & \phi'_{22} & \phi'_{32} & \phi'_{42} & -\Phi'_{12} & -\Phi'_{32} \\ f_{12} & f_{22} & f_{32} & f_{42} & -F_{12}/m & -F_{32}/m \\ g_{12} & g_{22} & g_{32} & g_{42} & -G_{12}/m & -G_{32}/m \end{vmatrix} = 0, \tag{40}$$

where

$$f_{n2} = \alpha^2\phi_{n2} + \phi''_{n2}, \qquad n = 1,2,3,4;$$

$$F_{n2} = \alpha^2\Phi_{n2} + \Phi''_{n2}, \qquad n = 1,3;$$

$$g_{n2} = \left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right]R\phi_{n2} - [(1-c)R - i3\alpha]\phi'_{n2} - \frac{i}{\alpha}\phi'''_{n2}, \quad n = 1,2,3,4;$$

$$G_{n2} = \left[m + \frac{1}{F(1-c)}\right]Rrm\Phi_{n2} - $$

$$-[(1-c)Rrm - i3\alpha]\Phi'_{n2} - \frac{i}{\alpha}\Phi'''_{n2}, \qquad n = 1,3.$$

5. *Solutions of the boundary value problem*

The purpose of our calculation from now on will be to find the function that the secular equation represents.

From (22), (23) and (38), it is clear that in the liquid

$$U = y, \qquad U' = 1, \quad (0 \leqslant y \leqslant 1),$$

and in the gas

$$U = 1 + m(y-1); \quad U' = m, \quad (1 \leqslant y \leqslant \infty). \tag{41}$$

The six solutions involved are, from (8), (11), (14), (17), (24), (25) and (32),

$$\phi_1(y) = e^{-\alpha y}, \tag{42}$$

$$\phi_2(y) = e^{\alpha y}, \tag{43}$$

$$\phi_3(y) = \frac{1}{\alpha}\left\{\int_1^y \exp\{\alpha(y-t_l)\}\zeta_1(t_l)\,dt_l - \int_1^y \exp\{-\alpha(y-t_l)\}\zeta_1(t_l)\,dt_l\right\}, \tag{44}$$

$$\phi_4(y) = \frac{1}{\alpha}\left\{\int_1^y \exp\{\alpha(y-t_l)\}\zeta_2(t_l)\,dt_l - \int_1^y \exp\{-\alpha(y-t_l)\}\zeta_2(t_l)\,dt_l\right\}, \tag{45}$$

$$\Phi_1(y) = e^{-\alpha y}, \tag{46}$$

$$\Phi_3(y) = \frac{1}{\alpha}\left\{\int_\infty^y \exp\{\alpha(y-t_g)\}\zeta_1(t_g)\,dt_g - \int_\infty^y \exp\{-\alpha(y-t_g)\}\zeta_1(t_g)\,dt_g\right\}, \tag{47}$$

where

$$\zeta_1(t_l) = h_1(z_l) = \frac{k}{i\pi}\int_{L_1} \exp(z_l t + \tfrac{1}{3}t^3)\,dt, \tag{48}$$

$$\zeta_2(t_l) = h_2(z_l) = -\frac{k^*}{i\pi}\int_{L_2} \exp(z_l t + \tfrac{1}{3}t^3)\,dt, \tag{49}$$

$$\zeta_1(t_g) = h_1(z_g) = \frac{k}{i\pi}\int_{L_1} \exp(z_g t + \tfrac{1}{3}t^3)\,dt, \tag{50}$$

$$z_l = \left[\frac{\alpha^2}{(\alpha R)^{2/3}} + i(\alpha R)^{1/3}(t_l - c)\right]\Bigg\}, \tag{51}$$

$$z_g = \left\{\frac{\alpha^2}{(\alpha Rrm^2)^{2/3}} + i(\alpha Rrm^2)^{1/3}\left[(t_g-1)+\frac{1}{m}(1-c)\right]\right\}, \tag{52}$$

and for clarity, instead of using y in the liquid and gas, the variables t_l and t_g were introduced. In (44) and (45), the lower limit of integration was taken as $y = 1$ for convenience in later calculations. (For an alternative refer to discussion following (55).) Later on, many expressions will have to be evaluated at $y = y_2 = 1$. With the choice made here, the calculation is simplified since several integrals vanish, i.e. ϕ_{32}, ϕ'_{32}, ϕ''_{32}, ϕ_{42}, ϕ'_{42}, ϕ''_{42} and ϕ'''_{42}. In (47) the lower limit was taken as infinity, because two of the boundary conditions required the solutions to vanish there: no other choice would have been satisfactory. The functions needed in (40) will now be written down:

$$\left.\begin{aligned}
&\phi_{11} = 1, && \phi_{21} = 1, && \\
&\phi'_{11} = -\alpha, && \phi'_{21} = \alpha, && \\
&\phi_{12} = e^{-\alpha}, && \phi_{22} = e^{\alpha}, && \Phi_{12} = e^{-\alpha}, \\
&\phi'_{12} = -\alpha e^{-\alpha}, && \phi'_{22} = \alpha e^{\alpha}, && \Phi'_{12} = -\alpha e^{-\alpha}, \\
&\phi''_{12} = \alpha^2 e^{-\alpha}, && \phi''_{22} = \alpha^2 e^{\alpha}, && \Phi''_{12} = \alpha^2 e^{-\alpha}, \\
&\phi'''_{12} = -\alpha^3 e^{-\alpha}, && \phi'''_{22} = \alpha^3 e^{\alpha}, && \Phi'''_{12} = -\alpha^3 e^{-\alpha}, \\
&\phi_{31} = \frac{1}{\alpha}\left\{\int_1^0 \exp(-\alpha t_l)\zeta_1(t_l)\,dt_l - \int_1^0 \exp(\alpha t_l)\zeta_1(t_l)\,dt_l\right\}, \\
&\phi'_{31} = \int_1^0 \exp(-\alpha t_l)\zeta_1(t_l)\,dt_l + \int_1^0 \exp(\alpha t_l)\zeta_1(t_l)\,dt_l, \\
&\phi_{32} = 0, \qquad \phi'_{32} = 0, \\
&\phi''_{32} = 2\zeta_1(t_l = 1) = 2h_1(z_l(1)], \\
&\phi'''_{32} = 2\zeta'_1(t_l = 1) = 2h'_1[z_l(1)]i(\alpha R)^{1/3}, \\
&\phi_{41} = \frac{1}{\alpha}\left\{\int_1^0 \exp(-\alpha t_l)\zeta_2(t_l)\,dt_l - \int_1^0 \exp(\alpha t_l)\zeta_2(t_l)\,dt_l\right\}, \\
&\phi'_{41} = \int_1^0 \exp(-\alpha t_l)\zeta_2(t_l)\,dt_l + \int_1^0 \exp(\alpha t_l)\zeta_2(t_l)\,dt_l, \\
&\phi_{42} = 0, \qquad \phi'_{42} = 0, \\
&\phi''_{42} = 2\zeta_2(t_l = 1) = 2h_2(z_l(1)], \\
&\phi'''_{42} = 2\zeta'_2(t_l = 1) = 2h'_2[z_l(1)]i(\alpha R)^{1/3}, \\
&\Phi_{32} = \frac{1}{\alpha}\left\{\int_\infty^1 \exp\{-\alpha(t_g-1)\}\zeta_1(t_g)\,dt_g - \int_\infty^1 \exp\{\alpha(t_g-1)\}\zeta_1(t_g)\,dt_g,\right. \\
&\Phi'_{32} = \int_\infty^1 \exp\{-\alpha(t_g-1)\}\zeta_1(t_g)\,dt_g + \int_\infty^1 \exp\{\alpha(t_g-1)\}\zeta_1(t_g)\,dt_g, \\
&\Phi''_{32} = 2\zeta_1(t_g = 1) + \alpha^2\Phi_{32} = 2h_1[z_g(1)] + \alpha^2\Phi_{32}, \\
&\Phi'''_{32} = 2\zeta'_1(t_g = 1) + \alpha^2\Phi'_{32} = 2h'_1[z_g(1)]i(\alpha Rrm^2)^{1/3}.
\end{aligned}\right\} \tag{53}$$

Here $z_l(1)$ and $z_g(1)$ denote the values for $t_l = 1$ and $t_g = 1$ respectively.

We shall next change the determinant of (40) into a form more convenient for calculation. The procedure is as follows:

(*a*) Divide the last row by R.

(*b*) Multiply the 3rd, 4th and 6th columns by $(\alpha R)^{1/2}/'\phi''_{32}$, $(\alpha R)^{1/2}/'\phi''_{42}$ and $(\alpha R r m^2)^{1/2}/'\Phi''_{32}$ respectively, where the accent to the left of a symbol means the leading term in the asymptotic expansion ($\alpha R \gg 1$) of the corresponding function.

(*c*) From (60), insert all the zeros for the terms that vanish.

(*d*) Multiply the 5th column by -1, and rearrange the columns so that column 5 becomes 3, 3 becomes 4, and 4 becomes 5.

The result is then

$$\begin{vmatrix} \Lambda_1 & V_1 & 0 & a_1 & b_1 & 0 \\ \Lambda_2 & V_2 & 0 & a_2 & b_2 & 0 \\ \Lambda_3 & V_3 & T_3 & 0 & 0 & c_3 \\ \Lambda_4 & V_4 & T_4 & 0 & 0 & c_4 \\ \Lambda_5 & V_5 & T_5 & a_5 & b_5 & c_5 \\ \Lambda_6 & V_6 & T_6 & a_6 & b_6 & c_6 \end{vmatrix} = 0, \tag{54}$$

where the first three columns involve the inviscid terms (except for the last row, which has terms of $O(1/R)$ and will be neglected in comparison with the terms kept in the calculation), and the last three columns involve the viscous solutions. The term 'viscous solutions' refers to any of ϕ_3, ϕ_4, Φ_3, or their derivatives, regardless of where they are evaluated. The meaning of the symbols in (54) is given by the following relationships:

$$\left.\begin{aligned} &\Lambda_1 = \phi_{11}, \quad \Lambda_2 = \phi'_{11}, \quad \Lambda_3 = \phi_{12}, \quad \Lambda_4 = \phi'_{12}, \quad \Lambda_5 = \alpha^2\phi_{12} + \phi''_{12}, \\ &\Lambda_6 = \left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right]\phi_{12} - (1-c)\phi'_{12} + i(3\alpha^2\phi'_{12} - \phi'''_{12})(\alpha R)^{-1}, \end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned} &V_1 = \phi_{21}, \quad V_2 = \phi'_{21}, \quad V_3 = \phi_{22}, \quad V_4 = \phi'_{22}, \quad V_5 = \alpha^2\phi_{22} + \phi''_{22}, \\ &V_6 = \left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1\right]\phi_{22} - (1-c)\phi'_{22} + i(3\alpha^2\phi'_{22} - \phi'''_{22})(\alpha R)^{-1}, \end{aligned}\right\} \tag{56}$$

$$\left.\begin{aligned} &T_3 = \Phi_{12}, \quad T_4 = \Phi'_{12}, \quad T_5 = m^{-1}(\alpha^2\Phi_{12} + \Phi''_{12}), \\ &T_6 = r\left\{\left[m + \frac{1}{F(1-c)}\right]\Phi_{12} - (1-c)\Phi'_{12}\right\} + im^{-1}(3\alpha^2\Phi'_{12} - \Phi'''_{12})(\alpha R)^{-1}, \end{aligned}\right\} \tag{57}$$

$$\left.\begin{aligned} &a_1 = (\alpha R)^{1/2}\phi_{31}/'\phi''_{32}, \quad a_2 = (\alpha R)^{1/2}\phi'_{31}/'\phi''_{32}, \\ &a_5 = (\alpha R)^{1/2}\phi''_{32}/'\phi''_{32}, \quad a_6 = -i(\alpha R)^{-1/2}\phi'''_{32}/'\phi''_{32}, \end{aligned}\right\} \tag{58}$$

$$\left.\begin{aligned} &b_1 = (\alpha R)^{1/2}\phi_{41}/'\phi''_{42}, \quad b_2 = (\alpha R)^{1/2}\phi'_{41}/'\phi''_{42}, \\ &b_5 = (\alpha R)^{1/2}\phi''_{42}/'\phi''_{42}, \quad b_6 = -i(\alpha R)^{-1/2}\phi'''_{42}/'\phi''_{42}, \end{aligned}\right\} \tag{59}$$

$$\left.\begin{aligned}
c_3 &= (\alpha Rrm^2)^{1/2}\Phi_{32}/'\Phi_{32}'', \quad c_4 = (\alpha Rrm^2)^{1/2}\Phi_{32}'/'\Phi_{32}'',\\
c_5 &= m^{-1}[\alpha^2(\alpha Rrm^2)^{1/2}\Phi_{32}/'\Phi_{32}''+(\alpha Rrm^2)^{1/2}\Phi_{32}''/'\Phi_{32}''],\\
c_6 &= m^{-1}\left[\left\{m+\frac{1}{F(1-c)}\right]mr(\alpha Rrm^2)^{1/2}\Phi_{32}/'\Phi_{32}''-\left[(1-c)mr-i\frac{3\alpha^2}{\alpha R}\right]\times\right.\\
&\qquad\left.\times(\alpha Rrm^2)^{1/2}\Phi_{32}'/'\Phi_{32}''-\frac{i}{\alpha R}(\alpha Rrm^2)^{1/2}\Phi_{32}'''/'\Phi_{32}''\right\}.
\end{aligned}\right\}\tag{60}$$

It is worth remarking that the lower limit for the integrals in (44) and (45) could alternatively have been taken as zero instead of unity. This would have lead to a fourth-order determinant instead of the sixth-order one in (54). Each element of the alternative determinant would have been not only more complicated than the one of (54), but the clear distinction between viscid and inviscid solutions (of which use is made for the calculations of the neutral stability lines) would not have been possible.

6. *Behaviour of the viscous solutions for $\alpha R \gg 1$*

Thus far, no restrictions of any kind have been made in the analysis. We will restrict the discussion to the case $\alpha R \gg 1$, this being sufficient to solve the problem. One of the major tasks is the evaluation of the integrals in (53). The only previous discussion of a similar integral is the one by Hopf (1914), who assumed $\alpha R \ll 1$; thus his discussion is not suited to the present study. In order to find out how the neutral stability curve (i.e. for given m, r, F and W) behaves for large values of the parameter αR, it will be sufficient to keep only the terms of highest order in αR in the asymptotic expansion of the pertinent functions. As will be shown later (following (71)), the case of interest for neutral stability (cf. (3)), is the one where $c = c_r < 1$. A detailed description of the method used for obtaining the integrals, the order of magnitude of the errors involved, and the calculation of ϕ_{31} as an example of the procedure used, is given in Appendix B of the author's thesis. If $\alpha R \gg 1$, from (51) and (52) it may be assumed that

$$\begin{aligned}
z_{l0} &\equiv z_l(0) = \alpha^2(\alpha R)^{-2/3}-i(\alpha R)^{1/3}c \sim -i(\alpha R)^{1/3}c,\\
z_{l1} &\equiv z_l(1) = \alpha^2(\alpha R)^{-2/3}+i(\alpha R)^{1/3}(1-c) \sim i(\alpha R)^{1/3}(1-c),\\
z_{g1} &\equiv z_g(1) = \alpha^2(\alpha Rrm^2)^{-2/3}+i(\alpha Rrm^2)^{1/3}(1-c)m^{-1} \sim i(\alpha Rrm^2)^{1/3}(1-c)m^{-1}.
\end{aligned}\tag{61}$$

Now, let

$$\left.\begin{aligned}
e^+ &= \exp[\tfrac{2}{3}(\alpha R)^{1/2}e^{i(5/4)\pi}(1-c)^{3/2}],\\
e^- &= \exp[-\tfrac{2}{3}(\alpha R)^{1/2}e^{i(5/4)\pi}(1-c)^{3/2}],\\
E &= \exp\{i\tfrac{2}{3}(\alpha R)^{1/2}[e^{-i(3/4)\pi}c^{3/2}-e^{i(3/4)\pi}(1-c)^{3/2}]\},
\end{aligned}\right\}\tag{62}$$

so that (58), (59) and (60) become

$$\left.\begin{aligned}
a_1 &= a_{11}(\alpha R)^{-1/2}E+O[(\alpha R)^{1/4}e^-],\\
a_2 &= a_{21}E+O[(\alpha R)^{1/4}e^-],\\
a_5 &= a_{51}(\alpha R)^{1/2}+a_{52}+O(\alpha R)^{-1/2},\\
a_6 &= a_{61}+a_{62}(\alpha R)^{-1/2}+O(\alpha R)^{-1},
\end{aligned}\right\}\tag{63}$$

$$\left.\begin{aligned}
b_1 &= b_{11}+b_{12}(\alpha R)^{-1/2}+O[(\alpha R)^{1/4}e^{+}],\\
b_2 &= b_{21}+b_{22}(\alpha R)^{-1/2}+O[(\alpha R)^{1/4}e^{+}],\\
b_5 &= b_{51}(\alpha R)^{1/2}+b_{52}+O[(\alpha R)^{-1/2}],\\
b_6 &= b_{61}+b_{62}(\alpha R)^{-1/2}+b_{63}(\alpha R)^{-1}+O[(\alpha R)^{-3/2}],
\end{aligned}\right\} \tag{64}$$

$$\left.\begin{aligned}
c_3 &= c_{31}(\alpha R)^{-1/2}+c_{32}(\alpha R)^{-1}+O[(\alpha R)^{-3/2}],\\
c_4 &= c_{41}+c_{42}(\alpha R)^{-1/2}+O[(\alpha R)^{-1}],\\
c_5 &= c_{51}(\alpha R)^{1/2}+c_{52}+c_{53}(\alpha R)^{-1/2}+O[(\alpha R)^{-1}],\\
c_6 &= c_{61}(\alpha R)^{-1/2}+c_{62}(\alpha R)^{-1}+O[(\alpha R)^{-3/2}],
\end{aligned}\right\} \tag{65}$$

where

$$\left.\begin{aligned}
a_{11} &= e^{i(3/4)\pi}c^{-5/4}(1-c)^{1/4}, & a_{52} &= -\tfrac{5}{48}e^{-i(\pi/4)}(1-c)^{-3/2},\\
a_{21}\, & U e^{-i(\pi/2)}c^{-3/4}(1-c)^{1/4}, & a_{61} &= -e^{-i(\pi/4)}(1-c)^{1/2},\\
a_{51} &= 1, & a_{62} &= \tfrac{7}{48}e^{i(\pi/2)}(1-c)^{-1},
\end{aligned}\right\} \tag{66}$$

$$\left.\begin{aligned}
b_{11} &= e^{-i(\pi/4)}\frac{\sinh\alpha}{\alpha(1-c)^{1/2}}, & b_{51} &= 1,\\
b_{12} &= e^{i(\pi/2)}\frac{\cosh\alpha}{1-c}, & b_{52} &= -e^{-i(\pi/4)}\tfrac{5}{48}(1-c)^{-3/2},\\
b_{21} &= -e^{-i(\pi/4)}\frac{\cosh\alpha}{(1-c)^{1/2}}, & b_{61} &= e^{-i(\pi/4)}(1-c)^{1/2},\\
b_{22} &= -e^{i(\pi/2)}\frac{\alpha\sinh\alpha}{1-c}, & b_{62} &= \tfrac{17}{48}e^{i(\pi/2)}(1-c)^{-1},\\
& & b_{63} &= -e^{i(\pi/4)}(\tfrac{35}{192}+\tfrac{385}{4608})(1-c)^{-5/2},
\end{aligned}\right\} \tag{67}$$

$$\left.\begin{aligned}
c_{31} &= -e^{i(\pi/2)}(rm^2)^{-1/2}\frac{m}{1-c}, & c_{51} &= r^{1/2}\\
c_{32} &= e^{i(\pi/4)}\alpha(rm^2)^{-1}\left(\frac{m}{1-c}\right)^{3/2}, & c_{52} &= -\tfrac{5}{4}e^{-i(\pi/4)}\left(\frac{m}{1-c}\right)^{3/2},\\
c_{41} &= -e^{-i(\pi/4)}\left(\frac{m}{1-c}\right)^{1/2}, & c_{53} &= -2e^{i(\pi/2)}\alpha^2(rm^2)^{-1/2}(1-c)^{-1},\\
c_{42} &= -\tfrac{41}{48}e^{i(\pi/2)}(rm^2)^{-1/2}\left(\frac{m}{1-c}\right)^{2}, & c_{61} &= -e^{i(\pi/2)}\frac{(rm^2)^{1/2}}{F(1-c)^2 m},\\
c_{62} &= e^{i(\pi/4)}\frac{1}{m}\left(\frac{m}{1-c}\right)^{1/2}\left\{\left[m+\frac{1}{F(1-c)}\right]\frac{\alpha}{1-c}-2\alpha^2+\frac{385}{4608}\left(\frac{m}{1-c}\right)^{2}\right\}.
\end{aligned}\right\} \tag{68}$$

The secular equation (54) can be expanded in terms of a sum of products of the viscid and inviscid terms. Equations (62) to (68) are then used for the purpose of determining the important terms to be kept in the final result.

7. *The eigenvalue problem for* $\alpha R \to \infty$

It can be shown that for $\alpha R \to \infty$, the secular equation reduces to

$$\mathscr{F}(\alpha, c, r, m, F, W) = 0,$$

or

$$(1-c)\alpha[g(\alpha, r, m)+r] - \left(\frac{\alpha^2}{W}+1-mr\right) - \frac{(1-r)}{F(1-c)} = 0, \tag{69}$$

which, when solved for c, gives*

$$c = 1 - \frac{(\alpha^2/W+1-rm)+\sqrt{\{(\alpha^2/W+1-rm)^2+4\alpha[g(\alpha,r,m)+r](1-r)/F\}}}{2\alpha[g(\alpha,r,m)+r]}, \tag{70}$$

where

$$g(\alpha, r, m) = \left[\frac{e^{\alpha}}{\sinh\alpha} + \frac{(m/r)^{1/2}}{\tanh\alpha} - 1\right]\frac{1}{1+(m/r)^{1/2}}, \tag{71}$$

the radicand being always positive, i.e. c is always real. This means that for $R \to \infty$, *the flow is neutrally stable*†. Also, since for cases of interest $rm < 1$, *c is always less than unity as $R \to \infty$*. This is quite an important conclusion, since for finite Reynolds numbers, the computation is different, depending on whether c is less or more than unity. It will therefore be assumed, in all the work for finite Reynolds number, that $c < 1$. The final calculations will bear out this assumption.

A different approach to the case of infinite Reynolds number would be to neglect viscosity at the outset, i.e. in the Orr–Sommerfeld equations. The differential equations then become of second order, and by relaxing the proper boundary conditions, the problem could again be solved. This has been done, and the equation obtained for c is the same as (70) except that the function $g(\alpha, r, m)$ is not given by (71) but by

$$g(\alpha, r, m) = \coth\alpha, \tag{72}$$

which agrees with (71) only when $m/r \gg 1$.

The limiting processes used to obtain (71) and (72) were different. Therefore it is not surprising that these results are not identical. From a physical point of view, the disagreement is inacceptable. Although no physical explanation of the discrepancy has been found as yet, the disagreement is immaterial from the present standpoint because for the fluid of interest, viz. a liquid–gas combination, $m/r \gg 1$ and the results agree.

8. *The case of finite Reynolds number*

The fundamental equations for the neutral stability curve in the αR-plane, for a given physical situation (i.e. for fixed gas–liquid density ratio r,

* The root with the negative sign in front of the radical can be proven to be an extraneous root introduced when solving (69).

† It is worth remarking that for the case of 'inviscid Couette flow' of a single fluid the Orr–Sommerfeld equation has no non-trivial solution, while for two layers of fluids there is only this one non-trivial solution for the eigenvalue problem.

liquid–gas viscosity ratio m, Froude number F, and Weber number W), are

$$\mathscr{G}(\alpha, c) - \mathscr{H}(\alpha, c) = 0, \qquad (\alpha R)^{1/2} = -\frac{\mathscr{H}(\alpha, c)}{\mathscr{F}(\alpha, c)}, \tag{73}$$

each point on the curve having a particular wave velocity. The functions $\mathscr{F}(\alpha, c)$, $\mathscr{G}(\alpha, c)$ and $\mathscr{H}(\alpha, c)$ are defined by

$$\mathscr{F}(\alpha, c) = \frac{1}{\sqrt{2}}\left\{2\alpha(1-c) + [e^{-\alpha}(\Lambda_6 - V_6) + 2T_6 \sinh\alpha]\left(\frac{m}{r}\right)^{1/2} + \right. \\ \left. + 2\sinh\alpha(T_6 - \Lambda_6)\right\}, \tag{74}$$

$$\begin{aligned}\mathscr{G}(\alpha, c) = &-\frac{2\alpha}{(1-c)^{1/2}}\left[\frac{5}{4}\left(\frac{m}{r}\right)^{1/2} m + \frac{17}{48} + \frac{1}{F(1-c)}\right] + \\ &+ 4\alpha^2 e^{-\alpha}\left(1 - \frac{1}{m}\right)\sinh\alpha\left(\frac{m}{r}\right)^{1/2}(1-c)^{1/2} - \\ &- [e^{-\alpha}(\Lambda_6 - V_6) + 2T_6\sinh\alpha]\left(\frac{m}{r}\right)^{1/2}\frac{1}{48(1-c)^{3/2}}\left[5 + 41\left(\frac{m}{r}\right)^{1/2}\right] + \\ &+ \frac{\alpha}{r(1-c)^{1/2}}[(V_6 - \Lambda_6)e^{-\alpha} + 2T_6\cosh\alpha] - \\ &- \frac{2\alpha}{(1-c)^{1/2}}(T_6 - \Lambda_6)\left[\frac{5}{4}\frac{m}{1-c}\left(\frac{m}{r}\right)^{1/2}\frac{\sinh\alpha}{\alpha} + \cosh\alpha\right] + \\ &+ 2\alpha\left(1 - \frac{1}{m}\right)(V_6 e^{-2\alpha} - \Lambda_6)\left(\frac{m}{r}\right)^{1/2}\frac{1}{(1-c)^{1/2}}\frac{\sinh\alpha}{\alpha},\end{aligned} \tag{75}$$

$$\mathscr{H}(\alpha, c) = c^{-1/2}\left\{-2\alpha^2(1-c) + \alpha[(\Lambda_6 + V_6)e^{-\alpha} - 2T_6\cosh\alpha]\left(\frac{m}{r}\right)^{1/2} - \right. \\ \left. - 2\alpha(T_6 - \Lambda_6)\cosh\alpha\right\}, \tag{76}$$

$$\Lambda_6 = \left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1 + (1-c)\alpha\right]e^{-\alpha}, \tag{77}$$

$$V_6 = \left[\frac{\alpha^2}{W} + \frac{1}{F(1-c)} + 1 - (1-c)\alpha\right]e^{\alpha}, \tag{78}$$

$$T_6 = \left[m + \frac{1}{F(1-c)} + (1-c)\alpha\right]re^{-\alpha}. \tag{79}$$

The solution of (73) for c as a function of α has to be obtained numerically*, because of the impossibility of solving explicitly for c or α in terms of the other. Once a pair of values of α and c is known, straightforward calculation leads to the value of $(\alpha R)^{1/2}$, it being possible to evaluate R immediately.

A digression is permissible at this point in order to point out the meaning of Reynolds number $R \equiv R_l$ in the present problem. Since the velocity profile is a linear function of distance, for given liquid physical properties R is proportional to the liquid flow rate, and therefore is a constant once the flow rate is specified.

* All the numerical work was carried out on an IBM card-programmed electronic calculator using an 8-digit floating decimal system.

Now that a method for determining the neutral stability curve has been described, it is necessary to decide which region, on either side of it, is stable or unstable. The calculations indicate, as is reasonable to expect, that for a disturbance of a given wavelength, the flow is stable for Reynolds numbers smaller than a Reynolds number R_n, corresponding to a neutral disturbance; and similarly the flow is unstable for $R > R_n$.

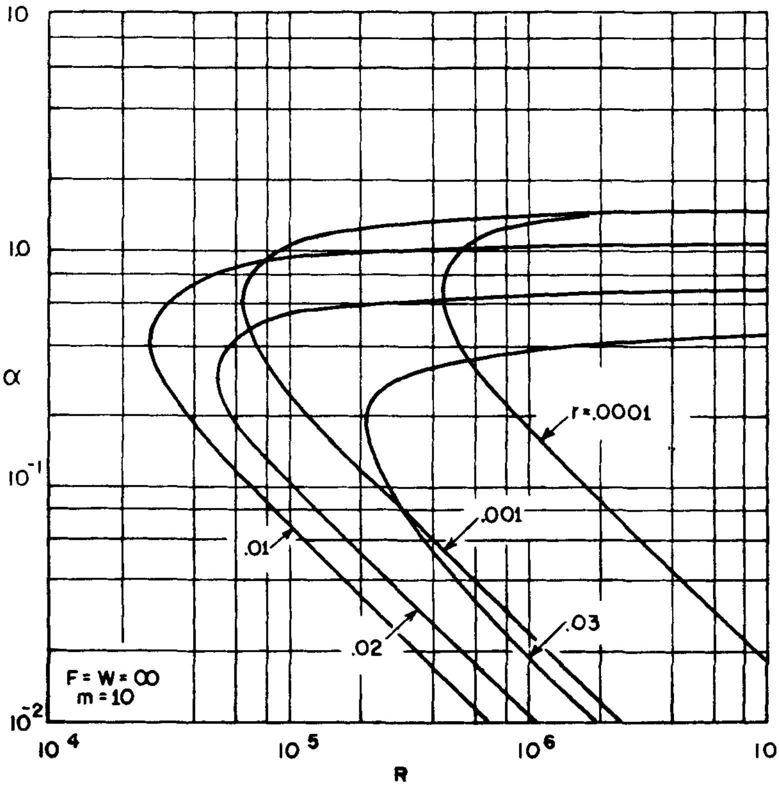

Figure 3. Neutral stability curves: wave number *vs* liquid Reynolds number. Gravity and surface tension forces are neglected. Liquid–gas viscosity ratio = 10.

The results obtained from the present analysis, of which figure 3 is a typical example with $F = W = 0$, indicate that the shape of the neutral stability curves in the αR-plane is similar to the boundary-layer case*. The effect of varying the gas–liquid density ratio r could be stabilizing or destabilizing, while increasing the liquid–gas viscosity ratio m always tends to stabilize the flow, provided m is not small compared with unity.

* The reason for obtaining only one solution to the problem, and not two as Lock (1954) found, is that in the present case each fluid is stable by itself, and the only reason for the existence of instability is the discontinuity in the physical properties of the fluids. In Lock's case each fluid is probably also unstable. On the other hand, in the present treatment instability regions which do not extend to $R = \infty$ have not necessarily been eliminated; this may occur when $c > 1$. The analysis would have to be extended to this case in order to investigate this possibility.

For $\alpha \to 0$, the equation for the lower branch of the neutral stability curve is

$$\alpha R = \text{const.}, \tag{80}$$

as can be derived analytically. The upper branch seems to have a nearly horizontal tangent for $R \to \infty$; this was obtained numerically.

The influences of non-vanishing gravity* or surface tension forces on the above results should be alike, since they enter as a sum in the secular relation (40). Figure 4 shows that for large Froude F and Weber numbers W,

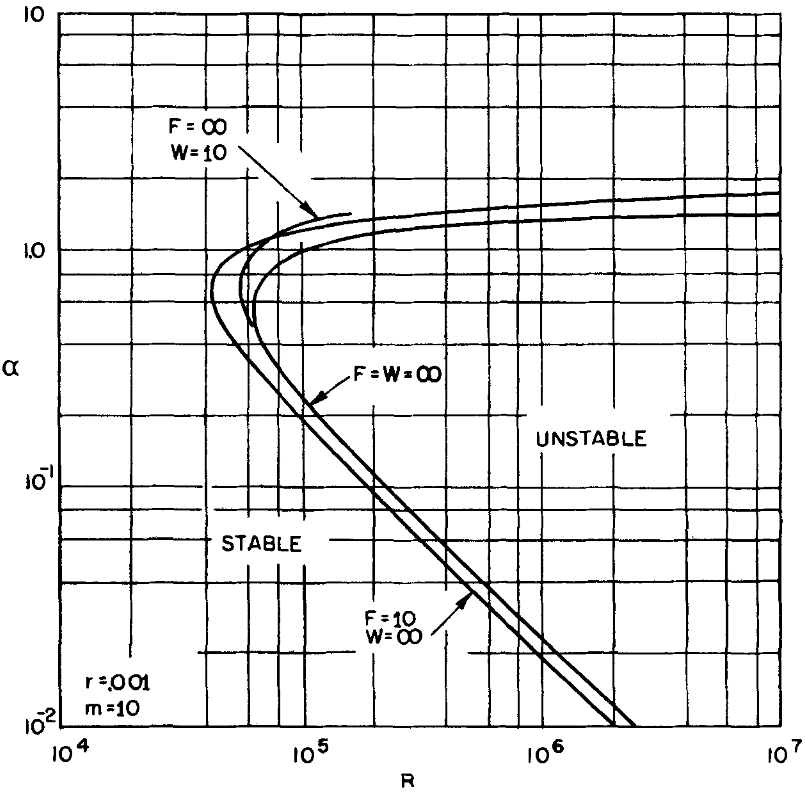

Figure 4. Effect of gravity and surface tension forces on neutral stability curves.

gravity and surface tension destabilize the flow. Obviously, surface tension effects are negligible when the disturbance frequency is small, but become very important at large frequencies. The effect on the neutral stability line is in this case to raise the upper branch so that its asymptote for $\alpha \gg 1$ is

$$\alpha/R = \text{const.}, \tag{81}$$

which can also be obtained analytically.

* The gravitational field was assumed to act downwards in the flow configuration indicated in figure 1.

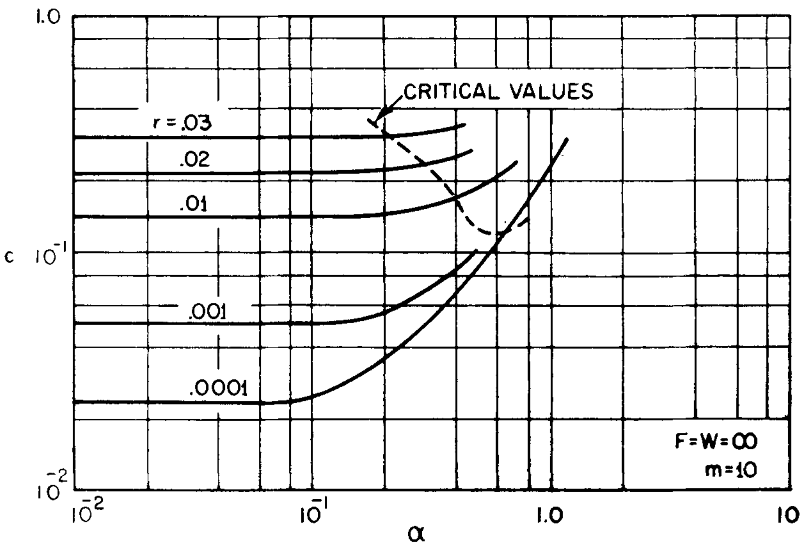

Figure 5. Wave velocity *vs* wave number for neutral stability. Gravity and surface tension forces are neglected. Liquid–gas viscosity ratio = 10.

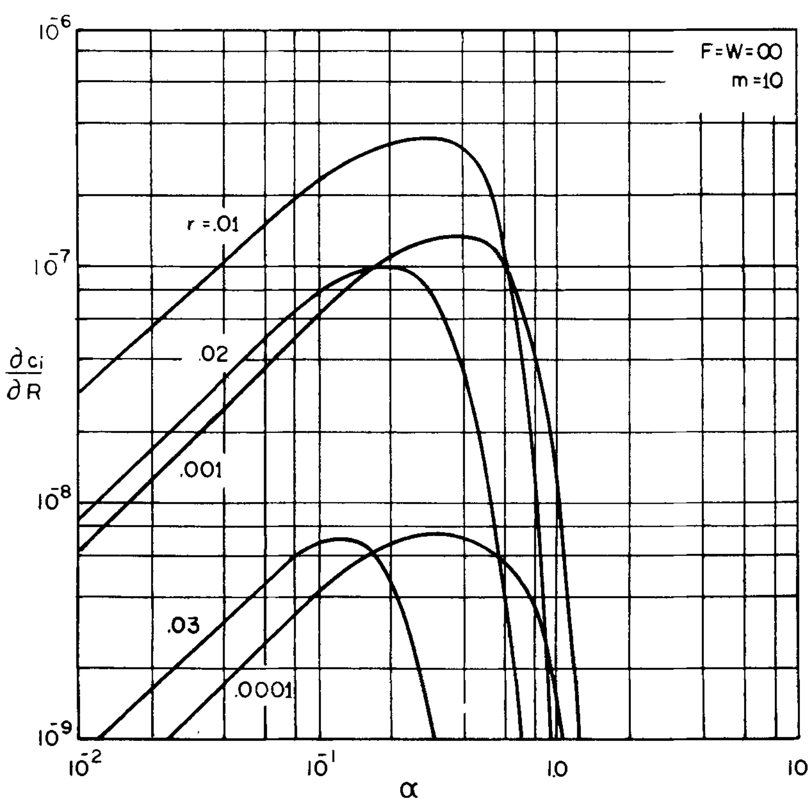

Figure 6. Rate of change of amplification factor with Reynolds number as a function of wave number and gas–liquid density ratio. Liquid–gas viscosity ratio = 10.

All the effects discussed thus far concern the stability of the flow, and the direction (left or right) in which the neutral stability curve moves in the αR-plane when the physical quantities are varied.

There are two or more items to be discussed before the results of our numerical calculations are summarized. They are (*a*) the behaviour of the magnitude of the wave velocity; (*b*) the amplification or damping rate of disturbances in the neighbourhood of the neutral stability line.

With respect to (*a*), it is enough to mention that all waves travel at speeds less than the velocity of the liquid–gas interface, a typical case being given in figure 5.

On the neutral stability curve, the imaginary part c_i of the complex velocity c vanishes. It is possible, nevertheless, to compute the rate of change of c_i with respect to Reynolds number. The important result is that this is always a positive quantity which, as a function of α (figure 6), has a peak near the critical Reynolds number. This means that the curves $c_i = \text{const.}$, in the αR-plane, would be packed close together when in the neighbourhood of the critical values of α and R.

Since the quantities of greatest interest are the parameters corresponding to the critical value of the Reynolds number, the large number of calculations which we have made may suitably be summarized as follows.

9. *Values of critical quantities and discussion of results. Comparison with experiments*

Since the physical case of interest is the one where gravity and surface tension forces are small, detailed calculations were carried out for $F = W = \infty$. The critical values are presented in figures 7 to 10, from which the following facts can be gathered:

(1) As m and $r \to 1$, the flow is completely stable.

(2) For a given gas–liquid density ratio r, an increase in the liquid–gas viscosity ratio m always increases the stability, and decreases the amplification or damping in the region away from the neutral stability curve.

(3) For very small r, the flow is completely stable.

(4) For a given m, there is always a value of r for which the flow is most unstable.

(5) As $r \to 0$, the wave number α tends to a value of 0·6 approximately, and the wave velocity c becomes 0·1 approximately.

(6) For the special case of air and water at a temperature of 100° C and a pressure of one atmosphere ($m \doteqdot 10$, $r \doteqdot 0{\cdot}001$), the value of the critical Reynolds number is 60 000.

(7) The critical conditions depend for given physical properties only on the liquid flow rate.

The significance of the results quoted as items (1) and (2) can be understood in terms of the known universal stability of plane Couette flow (from now on abbreviated as P.C.F.) between walls of arbitrary spacing. An explanation follows.

In the special case when the two fluids have the same density and viscosity (equivalent to the case of a single fluid), the velocity profile becomes a single straight line. Item (1) shows the flow then to be universally stable, in agreement with the known result for P.C.F.; a check on the analysis is thus obtained.

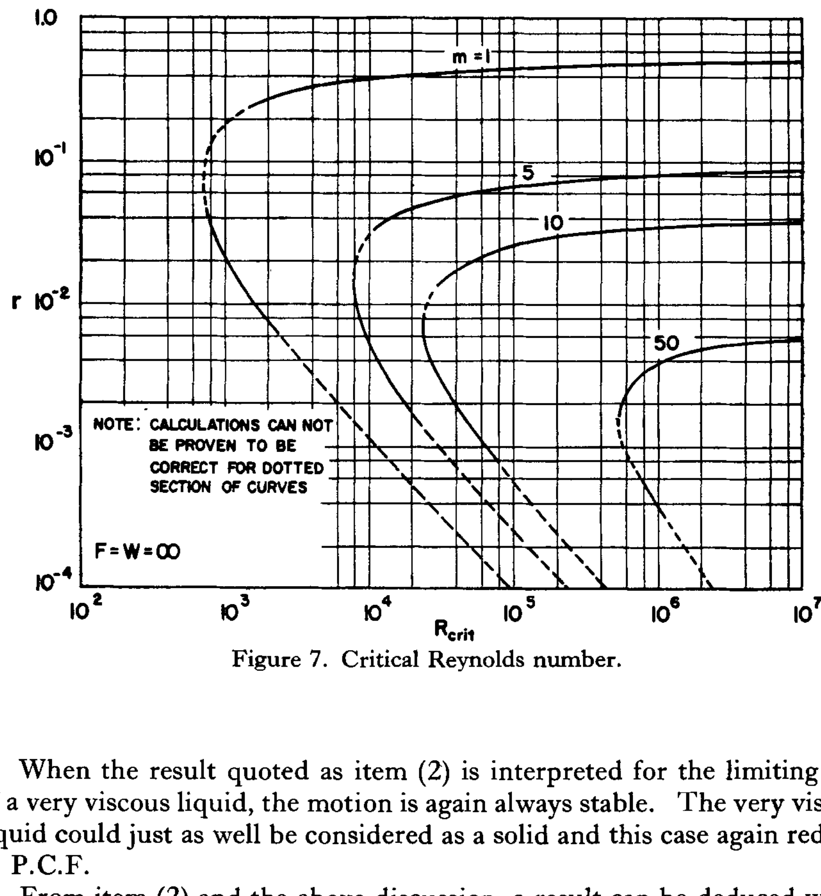

Figure 7. Critical Reynolds number.

When the result quoted as item (2) is interpreted for the limiting case of a very viscous liquid, the motion is again always stable. The very viscous liquid could just as well be considered as a solid and this case again reduces to P.C.F.

From item (2) and the above discussion, a result can be deduced which has not been obtained directly by any calculation. This is that, for an arbitrary density ratio, the flow is completely stabilized when $m \to 0$. The reason for stabilization is that this limiting case of the flow occurs when the gas becomes so viscous that it could be replaced by a solid wall, which again reduces to P.C.F. between two walls at a finite spacing.

Therefore, in figure 7 there would exist a curve for some small value of m that would be farthest to the left, and for smaller values of m the curves would again be displaced more and more to the right as the liquid–gas viscosity ratio is decreased.

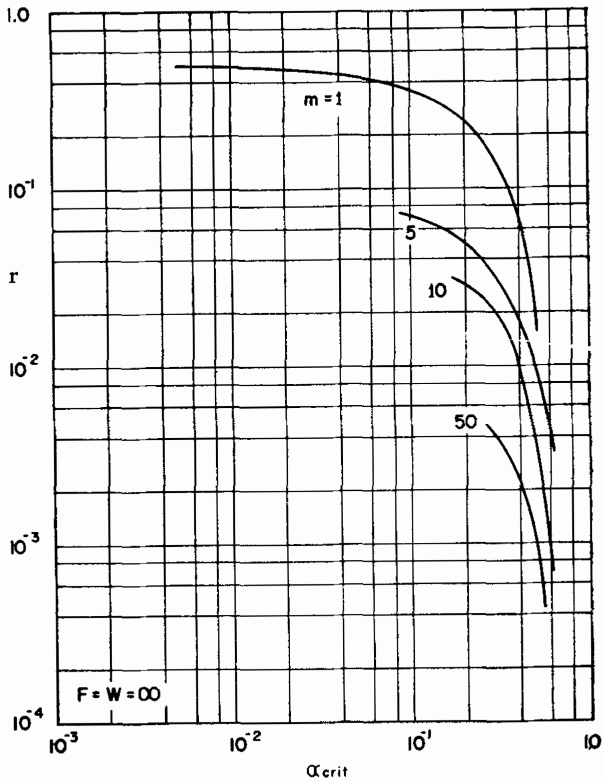

Figure 8. Critical wave number.

Items (5) and (6) will now be compared with some experimental observations. Most experiments with liquid films have been carried out in horizontal round tubes where the liquid flows along the inner surface dragged by a high-speed turbulent gas. It should be kept in mind that while the gas layer is of infinite thickness in the theoretical model, the ratio of laminar sublayer thickness in the gas to liquid film thickness in the experiments was of the order of unity.

The experimental neutral wavelength λ observed by Knuth (1954) for all liquid flow rates was about 10 film thicknesses. Considering that $\alpha = 2\pi/\lambda$, Knuth's findings check with the critical value of item (5). Since the wavelengths in the neighbourhood of the critical value are the most amplified, these might be the only ones visible in an experiment. Knuth might have observed these values, which, although seemingly neutral, could have been slightly amplified.

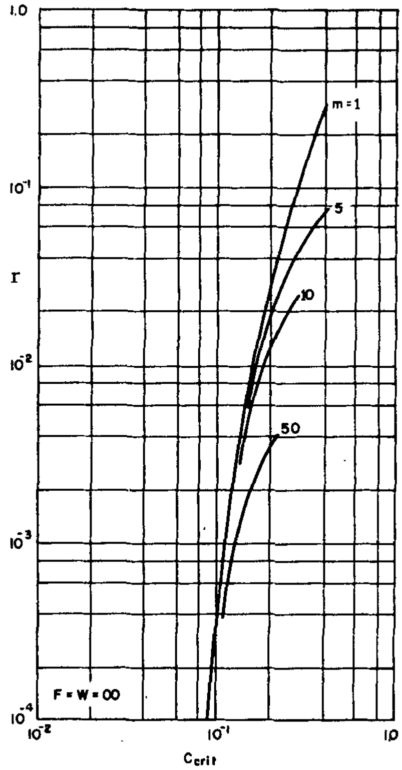

Figure 9. Critical wave velocity.

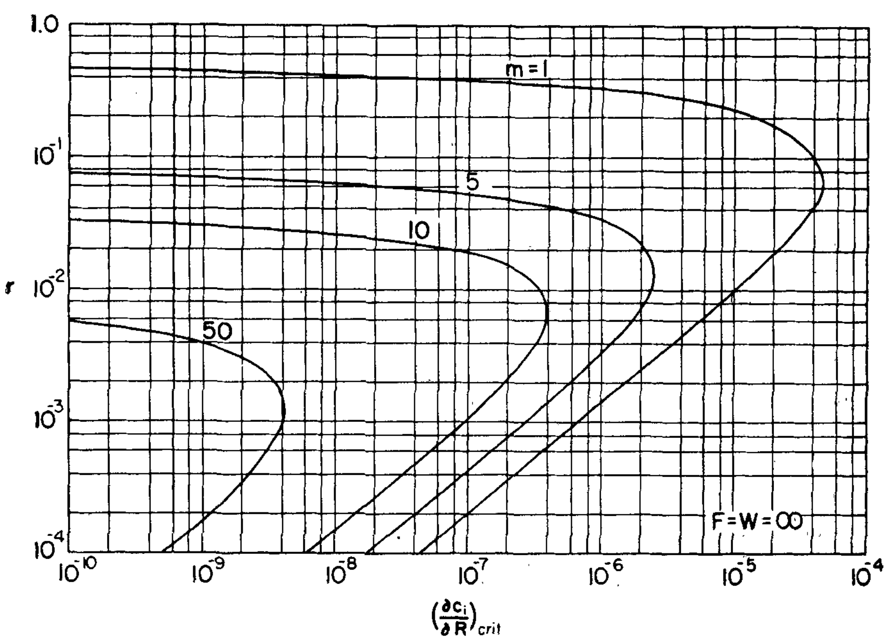

Figure 10. Critical values of the rate of change of amplification factor with Reynolds number.

Item (6) implies that for a liquid film 0·005 in. thick, the critical liquid–gas interface velocity is 100 ft/sec, which is one order of magnitude larger than the values obtained from experiments in liquid films. This discrepancy would seem to indicate that the observed instability is not simply laminar instability of uniform shearing motion. The fact that the computed critical Reynolds number is much higher than the experimental values could be due to the fact that the velocity profiles of both fluids in the analysis were assumed to be straight lines. If curved profiles were used it is conceivable that the critical Reynolds number could decrease. This possibility is suggested by the change in the value of critical Reynolds

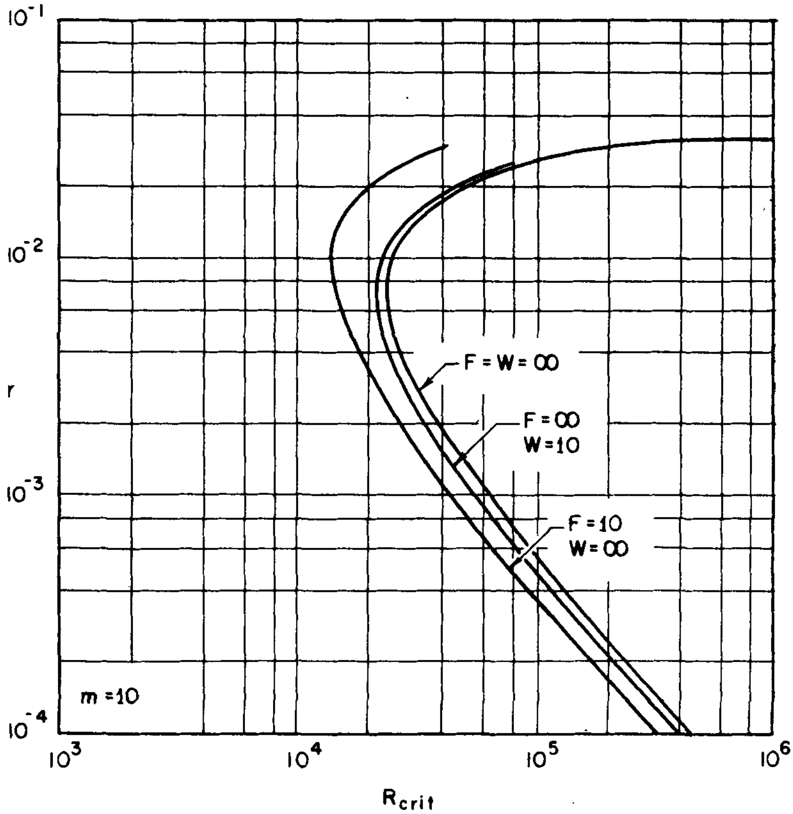

Figure 11. Influence of gravity and surface tension forces on critical Reynolds number.

number when going from plane Couette ($R_{\text{crit}} = \infty$) to plane Poiseuille flow ($R_{\text{crit}} = 11\,560$, based on the maximum velocity and width of the channel). Curvature in the velocity profile of the liquid could exist in the case of laminar flow when there is a pressure gradient in the flow direction. One of the things that has not been accounted for in the present or suggested analysis is the effect of turbulence of the gas stream, which also possibly influences the stability of the flow.

Gravity and surface tension have a destabilizing effect (see figures 11 and 12), as remarked in the discussion in § 8. This influence of surface tension was observed by Kinney, Abramson & Sloop (1952, p. 10), the relative change in the critical Reynolds number being of the same order of magnitude (within a factor of at most 2) as the value found here analytically.

Item (7) agrees with the experimental fact reported by Knuth that the inception point of instability is a function of the liquid flow rate, and independent of the gas flow rate.

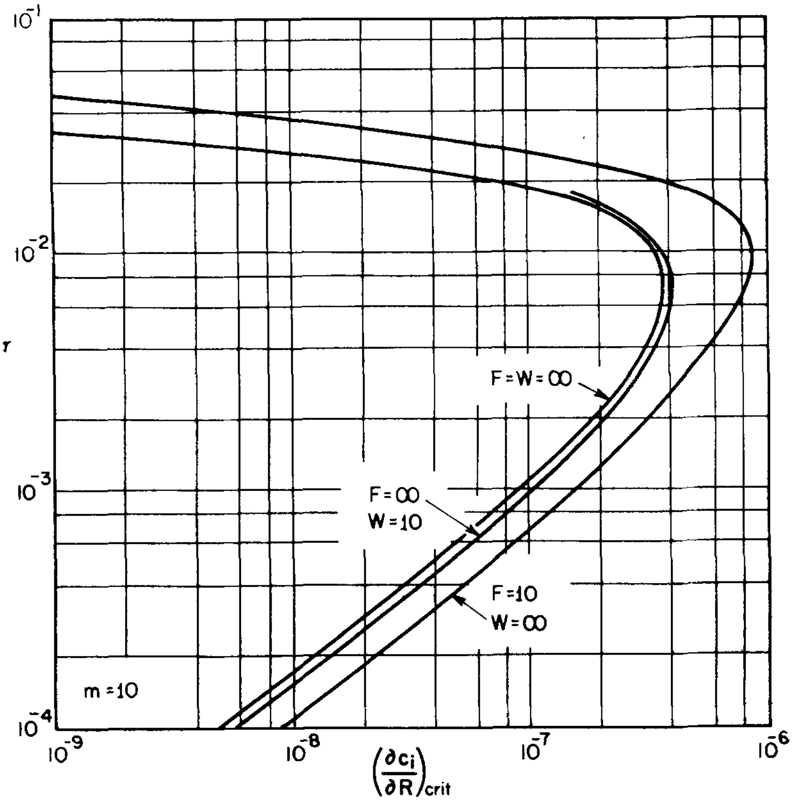

Figure 12. Influence of gravity and surface tension forces on critical rate of change of amplification factor with Reynolds number.

Before concluding, it will be helpful to try to gain some physical insight into the stability problem by looking at the energy of the disturbed motion. This will be done in Part III.

10. *Accuracy of calculations*

The accuracy of our calculations, based upon the viscous functions given in § 6, Part II, is restricted by the fact that αR was assumed large. This was interpreted as meaning that in (61) the right-hand members are good approximations to the left-hand members, and permitted the use of asymptotic methods. Terms of $O[(\alpha R)^{-1}]$ were neglected when compared with terms of $O[(\alpha R)^{-1/2}]$ in the secular equation. A check on the assumption mentioned was done for each calculation of a point on the neutral stability curve. The result of this checking showed that the assumption was incorrect for small values of r, shown dotted on figure 7 only. Nevertheless, the trends in that region are probably correct. This is inferred by comparing the dotted sections with the curve for $m = 50$, which is valid over almost all of the regions shown: they seem to form a reasonable family.

For small values of r, the last of equations (61) for z_{g1} should be approximated by the first term and not by the second, as for larger values of r. If this innovation were made, a valid calculation for the dotted part of figure 7 would then be possible.

III. The Reynolds shearing stress

A different way of looking at the stability problem, due to Lin (1954), consists of following the history of the disturbance energy, which, for damping or amplification, changes as a result of the action of the Reynolds shearing stress τ_s*. It is then enlightening to know its distribution across the stream.

Foote & Lin (1950) have shown that

$$\tau_s = -\rho\overline{uv} = -\frac{\rho\alpha}{4i}\exp\{2\alpha c_i t\}(\phi\bar{\phi}' - \bar{\phi}\phi'), \tag{82}$$

and

$$\frac{d\tau_s}{dy} = -\frac{\rho\alpha}{4i}\exp\{2\alpha c_i t\}\frac{d}{dy}(\phi\bar{\phi}' - \bar{\phi}\phi'), \tag{83}$$

where $\overline{uv}$ is the distance or time average of the product uv, and $\bar{\phi}$ and $\bar{\phi}'$ dicate complex conjugates of ϕ and ϕ'.

At the interface, for the viscous case,

$$\overline{u_l v_l} = \overline{u_g v_g} \tag{84}$$

$$(\tau_{sl}/\tau_{sg})_{\text{interface}} = \rho_l/\rho_g = 1/r. \tag{85}$$

The inviscid case

The amplitude functions ϕ and Φ for the inviscid case are real functions. Therefore (82) shows that

$$(\phi\bar{\phi}' - \bar{\phi}\phi') = \text{const.} = 0, \tag{86}$$

from which it follows that the Reynolds stresses are zero across the flow. This means that there is no mechanism for transferring energy between the basic flow and the disturbance, i.e. any disturbance will just subsist, without damping or amplification.

As can be seen, the method of this section is extremely useful, since the important result of § 7 regarding stability has now been rederived in a few lines without any calculation.

* The work done per unit volume per unit time by the basic flow is $\tau\, dU/dy$ (Schlichting 1950), and converts energy from the basic flow into the disturbance when $dU/dy > 0$.

The viscous case

We will start the study of this case by showing that the Reynolds stress is continuous across the layer where $U(y) = c_r$. The Orr–Sommerfeld equation (4) can be rewritten as (c is complex)

$$\phi'' - \alpha^2\phi = \frac{U''}{U-c}\phi - \frac{i}{\alpha R(U-c)}(\phi^{\mathrm{iv}} - 2\alpha^2\phi'' + \alpha^4\phi), \tag{87}$$

which, on multiplying by $\bar{\phi}$, subtracting its complex conjugate and regrouping yields

$$\frac{d}{dy}(\bar{\phi}\phi' - \bar{\phi}'\phi) = \frac{2c_i U''|\phi|^2}{|U-c|^2} - \frac{i}{\alpha R}\left(\frac{\phi^{\mathrm{iv}}\bar{\phi} - 2\alpha^2\phi''\bar{\phi} + \alpha^4\phi\bar{\phi}}{U-c} + \right.$$
$$\left. + \frac{\bar{\phi}^{\mathrm{iv}}\phi - 2\alpha^2\bar{\phi}''\phi + \alpha^4\phi\bar{\phi}}{U-\bar{c}}\right). \tag{88}$$

Equation (88) can now be introduced into (83), and since $U'' = 0$, we have

$$\frac{d\tau_s}{dy} = \frac{\rho}{4R}e^{2\alpha c_i t}\left(\frac{\phi^{\mathrm{iv}}\bar{\phi} - 2\alpha^2\phi''\bar{\phi} + \alpha^4\phi\bar{\phi}}{U-c} + \frac{\bar{\phi}^{\mathrm{iv}}\phi - 2\alpha^2\bar{\phi}''\phi + \alpha^4\phi\bar{\phi}}{U-\bar{c}}\right), \tag{89}$$

which can be integrated across the layer where $U = c$, and remembering that in our case $U = y$,

$$\int d\tau_s = (\tau_s)_{y=c+0} - (\tau_s)_{y=c-0} = [\tau_s]$$
$$= \frac{\rho}{4R}e^{2\alpha c_i t}\lim_{\epsilon\to 0}\{[\phi^{\mathrm{iv}}\bar{\phi} - 2\alpha^2\phi''\bar{\phi} + \alpha^4\phi\bar{\phi}]_{y=c}[\log\epsilon - \log(-\epsilon)] +$$
$$+ [\bar{\phi}^{\mathrm{iv}}\phi - 2\alpha^2\bar{\phi}''\phi + \alpha^4\phi\bar{\phi}]_{y=c}[\log(c-\bar{c}+\epsilon) - \log(c-\bar{c}-\epsilon)]\}. \tag{90}$$

Since $\qquad \log\epsilon - \log(-\epsilon) = \pm i\pi(2n+1), \qquad n = 0, 1, 2, \ldots,$

equation (90), with $c_i = 0$, gives for the jump in Reynolds stress $[\tau_s]$ across the layer where $U = c$

$$[\tau_s] = \frac{\rho}{4R}[\bar{\phi}^{\mathrm{iv}}\phi + \phi^{\mathrm{iv}}\bar{\phi} - 2\alpha^2(\phi''\bar{\phi} + \bar{\phi}''\phi) + 2\alpha^2\phi\bar{\phi}][\pm i\pi(2n+1)]. \tag{91}$$

where a particular n should be chosen for the branch of the logarithm being used. (Note that the quantity in the first bracket on the right-hand side of (91) is real, and the quantity in the second bracket is purely imaginary.) Since $[\tau_s]$ must be real, and $\rho/4R \neq 0$, we have at $y = c$

$$\phi^{\mathrm{iv}}\bar{\phi} + \bar{\phi}^{\mathrm{iv}}\phi - 2\alpha^2(\phi''\bar{\phi} + \bar{\phi}''\phi) + 2\alpha^4\phi\bar{\phi} = 0. \tag{92}$$

Therefore $\qquad [\tau_s] = 0 \quad \text{at} \quad y = c_r = c. \tag{93}$

Lin (1954) has shown that the Reynolds stress in a viscous fluid grows positively and very rapidly with distance away from the wall in a very thin layer, and then stays about constant.

We now have enough information to build the complete picture of stress distribution for the viscous case. Starting at the wall, the Reynolds stress is zero, and as we proceed outwards it grows at a rapid rate. It then levels off and since viscosity is not very important, (86) will be almost satisfied (i.e. $\tau_s \doteqdot$ const.). There is no jump across the layer where $U = c$. As we get to the liquid–gas interface, the stress is discontinuous, the ratio of the values on both sides of the discontinuity being given by (85). The stress on the gas side cannot be zero, since by (84) this would mean that the stress in the liquid is zero. Far away in the gas stream, the Reynolds stress must vanish. Figure 13 shows qualitatively this stress distribution.

Lin (1954) found an important relationship in the theory of hydrodynamic stability in a very simple way. The principle used by him was to equate the stress in the fluid adjacent to the wall computed by two different methods, starting from the wall in one case, and from the main stream in the other. This method has not yet been made successful here, the reason being that the magnitude of the jump of stress across the interface is unknown, only the ratio being known. In order to compute the jump, one would have first to compute the stress at the interface, on the gas side, by calculating the function Φ. This would destroy the simplicity of Lin's method.

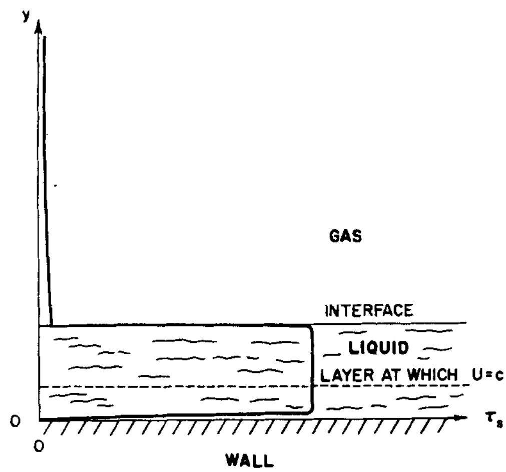

Figure 13. Distribution of Reynolds stress for a neutral oscillation.

Therefore, looking at the Reynolds stress in the present case was not as fruitful as in Lin's case. It would nevertheless be interesting to find the quantitative distribution of stress for a self-excited disturbance and for a neutral disturbance. This would show whether most of the energy input into the disturbance comes from the gas or the liquid. From the functions presented in this paper, it would be possible to calculate the Reynolds stress for a neutral disturbance. The case of a self-excited disturbance is much more complicated, and the necessary amount of numerical work, as envisaged at the present time, is prohibitive.

IV. Concluding remarks

The discrepancy between experiments on the stability of liquid films and the present theory is confined to the fact that the theoretical value of the critical Reynolds number is larger than the experimental one, all other quantities being in agreement. The results obtained here seem to indicate that a study of the hydrodynamic stability of two fluids with curved velocity profiles would be worth-while. Such a study might definitely settle the question whether or not the large-scale disturbances observed in liquid-film cooling experiments are due to laminar instability.

The distribution of the Reynolds stress across the stream remains a problem of great importance to the physical understanding of hydrodynamic stability in the present example.

The model chosen for the analysis, although the simplest possible, has yielded a number of new and interesting results. The most important conclusion to be drawn from this investigation is that a discontinuity of viscosity or density has a destabilizing effect on uniform shearing motion.

This paper is based on the thesis submitted by the author in partial fulfilment of the requirements for the degree of Doctor of Philosophy at the California Institute of Technology, June 1955. The author wishes to express his gratitude to Professor W. D. Rannie, for suggesting this problem and for his critical guidance during the course of this investigation, and to Professor C. C. Lin for his very helpful suggestions and personal interest. The author is also grateful to the Daniel and Florence Guggenheim Foundation and to the California Institute of Technology for making possible his continuation of graduate study through the grants, respectively, of a Fellowship in Jet Propulsion and a Graduate Tuition Scholarship during the 1953–54 and 1954–55 academic years.

References

Copson, E. T. 1935 *An Introduction to the Theory of Functions of a Complex Variable*, 1st Ed. Oxford University Press.

Feldman, S. 1955 *On the hydrodynamic stability of two viscous incompressible fluids in parallel uniform shearing motion.* Ph. D. Thesis, California Institute of Technology.

Foote, J. R. & Lin, C. C. 1950 *Quart. Appl. Math.* **8**, 265.

Goldstein, S. 1931 *Proc. Roy. Soc.* A, **132**, 524.

Hopf, L. 1914 *Ann. Physik* **44**, 11.

Kinney, G. R., Abramson, A. E. & Sloop, J. L. 1952 *Nat. Adv. Comm. Aero., Wash., Rep.* no. 1087.

Knuth, E. H. 1954 *Jet Propulsion* **24**, 359.

Lamb, H. 1932 *Hydrodynamics*, 6th Ed., p. 456. New York : Dover Publications.

Lin, C. C. 1945 *Quart. Appl. Math.* **3**, 117.

Lin, C. C. 1954 *Proc. Nat. Acad. Sci.* **40**, 741.

Lock, R. C. 1954 *Proc. Camb. Phil. Soc.* **50**, 105.

Morse, P. M. & Feshbach, H. 1953 *Methods of Theoretical Physics*, Vol. 1. New York : McGraw-Hill Book Co., Inc.

Orr, W. Mc. F. 1906 *Proc. Roy. Irish Acad.* **27**, 9.

Rayleigh, Lord 1887 *Scientific Papers*, Vol. 3, 17.

Schlichting, H. 1950 *Nat. Adv. Comm. Aero., Wash., Tech. Mem.* no. 1265.

Sommerfeld, A. 1909 *Atti del IV Congresso Internazionale dei Matematici*, Vol. 3, p. 116, Roma.

Taylor, G. I. 1931 *Proc. Roy. Soc.* A, **132**, 499.

Thomas, L. H. 1953 *Phys. Rev.* **91**, 780.

York, J. L., Stubbs, H. E. & Teck, M. R. 1953 *Trans. Amer. Soc. Mech. Engrs.* **75**, 1279.

Zondek, B. & Thomas, L. H. 1953 *Phys. Rev.* **90**, 738.